INJECTIVE BANACH SPACES OF TYPE $C(T)$ †

BY

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ABSTRACT

A Banach space X is a P_λ-space if whenever X is isometrically embedded in another Banach space Y there is a projection of Y onto X with norm at most λ. $C(T)$ denotes the Banach space of continuous real-valued functions on the compact Hausdorff space T. T satisfies the countable chain condition (CCC) if every family of disjoint non-empty open sets in T is countable. T is extremally disconnected if the closure of every open set in T is open. The main result is that if T satisfies the CCC and $C(T)$ is a P_λ-space, then T is the union of an open dense extremally disconnected subset and a complementary closed set T_A such that $C(T_A)$ is a $P_{\lambda-1}$-space.

1. Introduction

A Banach space X is injective if it has the following Hahn-Banach type extension property: every bounded linear operator $L : W \to X$ from a subspace $W \subset Z$ of a Banach space Z can be extended to a bounded linear operator $L' : Z \to X$. Goodner [7] introduced a family of Banach spaces coinciding with the family of injective spaces: for any $\lambda \geqq 1$, a Banach space X is a P_λ-space if, whenever X is isometrically embedded in another Banach space, there is a projection onto the image of X with norm not larger than λ. A Banach space is injective if and only if it is a P_λ-space for some $\lambda \geqq 1$ (Day [5, p. 94]).

Goodner [7], Nachbin [11] and Kelley [9] characterized the P_1-spaces: a Banach space is a P_1-space if and only if it is isometrically isomorphic to the Banach space of continuous functions on an extremally disconnected compact Hausdorff space. A topological space is *extremally disconnected* if the closure of any open set is open.

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The letter T will always denote a compact Hausdorff space and $C(T)$ will denote the Banach space of continuous real-valued functions on T with the supremum norm.

In this paper we examine the Banach spaces of type $C(T)$ which are P_λ-spaces for $\lambda > 1$. We continue the work of Amir ([1], [2], and [3]; also see [4], [8]) who has shown the following: *if $C(T)$ is an injective Banach space, then T contains an open dense extremally disconnected subset.*

To state our main theorem we define the Amir boundary of an arbitrary compact Hausdorff space. Let T be a compact Hausdorff space. Since the union of any family of open extremally disconnected sets is open and extremally disconnected, T contains a unique maximal open extremally disconnected subset. We define the *Amir boundary* of T (which will be denoted T_A) to be the complement of this maximal set. Thus T has a unique decomposition into an open extremally disconnected set and a complementary closed set T_A. Amir's result says that if $C(T)$ is an injective Banach space, then T_A is a nowhere dense set.

A compact Hausdorff space T satisfies the *countable chain condition* (abbreviated CCC) if any disjoint family of non-empty open sets in T is countable.

Our main result and its corollaries follow.

THEOREM 1.1. *Suppose T satisfies the* CCC. *If $C(T)$ is a P_λ-space, then the Amir boundary T_A of T also satisfies the* CCC *and $C(T_A)$ is a $P_{\lambda-1}$-space.*

COROLLARY 1.2. *Suppose T satisfies the* CCC. *If $C(T)$ is a P_λ-space for $\lambda < 3$ then the Amir boundary T_A of T is extremally disconnected.*

COROLLARY 1.3. *Suppose $C(T)$ is injective and T satisfies the* CCC. *Then T is a finite union of extremally disconnected spaces. More specifically, there is a decreasing finite sequence $T_0, T_1, \cdots, T_n$ of closed subsets of T beginning with $T_0 = T$ such that $T_i - T_{i+1}$ is an open dense extremally disconnected subset of T_i for each $i = 0, \cdots, n-1$ and T_n is extremally disconnected.*

A topological space is *totally disconnected* if it has a basis of closed and open sets.

COROLLARY 1.4. *Suppose $C(T)$ is injective and T satisfies the* CCC. *Then T is totally disconnected.*

These results are obtained by examining implications of the existence of averaging projections for Gleason maps (see Section 2 for the definitions of these terms). These implications are contained in Proposition 2.1. Section 2 contains a series of lemmas leading to the proof of this key proposition. Proposition 2.1 is used in Section 3 to prove the main theorem and its corollaries. Section 4 contains remarks about the CCC and three open problems.

2. Definitions, notation, and the key proposition

In the next two paragraphs we establish some notation and terminology. Let $\phi : S \to T$ be an onto continuous map where, as always, S and T are compact Hausdorff spaces. Then ϕ determines an isometric embedding $\phi^0(f) = f \circ \phi$ for each $f \in C(T)$. The range of ϕ^0 is the subalgebra $A_\phi = \{f \in C(S) : f$ is constant on each set $\phi^{-1}(t)$ for $t \in T\}$. An *averaging projection for* ϕ is a bounded linear projection $P : C(S) \to A_\phi$, that is, $P(f) = f$ for $f \in A_\phi$.

A continuous function $\phi : S \to T$ is *irreducible* if it is onto but no proper closed subset of S maps onto T. A *Gleason map* for a space T is an irreducible map $\phi : S \to T$ whose domain S is extremally disconnected. Every compact Hausdorff space has a Gleason map (Gleason [6]). For any space T, a Gleason map $\phi : S \to T$ determines an isometry $\phi^0 : C(T) \to A_\phi$ of $C(T)$ onto the subspace A_ϕ of the P_1-space $C(S)$. The Banach space $C(T)$ is a P_λ-space if and only if some Gleason map for T has an averaging projection with norm not larger than λ.

Theorem 1.1 will follow easily from the next proposition.

PROPOSITION 2.1. *Let $\phi : S \to T$ be a Gleason map which has an averaging projection $P : C(S) \to A_\phi$. Let T_A be the Amir boundary of T, let $S_1 = \phi^{-1}(T_A)$, and let $\phi_1 = \phi \mid S_1 : S_1 \to T_A$ be the restriction of ϕ to S_1. Then*

(a) *$\phi_1 : S_1 \to T_A$ has an averaging projection $P_1 : C(S_1) \to A_{\phi_1}$ with $\| P_1 \| \leqq \| P \| - 1$,*

(b) *if S satisfies the CCC, then so does S_1 and thus S_1 is extremally disconnected.*

The balance of this section contains several lemmas leading to the proof of Proposition 2.1. A set containing more than one point is called *plural*.

LEMMA 2.2. *Let $\phi : S \to T$ be an onto continuous map. Let $P_\phi = \{t \in T : \phi^{-1}(t)$ is plural$\}$, $J = \bar{P}_\phi$, $S_1 = \phi^{-1}(J)$ and let $\phi_1 = \phi \mid S_1 : S_1 \to J$. Let $P : C(S) \to A_\phi$ be an averaging projection for ϕ. Then the following are true.*

(a) *Define $P_1 : C(S_1) \to A_{\phi_1}$ by $P_1(f) = P(f') \mid S_1$ where $f' \in C(S)$ is any function with $f' \mid S_1 = f$. Then P_1 is an averaging projection for $\phi_1 : S_1 \to J$.*

(b) *Define $E : C(S_1) \to C(S)$ by $E(f) = f' - P(f')$ for $f \in C(S_1)$ where $f' \in C(S)$ is any function with $f' \mid S_1 = f$. Then E is a well-defined linear operator.*

(c) *If $\phi : S \to T$ is irreducible, then the averaging projection P_1 defined in* (a) *satisfies* $\| P_1 \| \leqq \| P \| - 1$.

PROOF. If $f' \in C(S)$ and $f'' \in C(S)$ are two extensions of $f \in C(S_1)$, then $f' - f'' \in A_\phi$ and thus $P(f' - f'') = f' - f''$. Therefore $P(f') \,|\, S_1 - P(f'') \,|\, S_1 = (f' - f'') \,|\, S_1 = 0$ and P_1 is well-defined. If $f \in A_{\phi_1}$ and f' is any extension of f, then $f' \in A_\phi$ since all plural sets $\phi^{-1}(t)$ are in S_1. Thus $P_1(f) = P(f') \,|\, S_1 = f' \,|\, S_1 = f$ for $f \in A_{\phi_1}$. So P_1 is an averaging projection for ϕ_1. The operator E in part (b) is well-defined since for two extensions f' and f'' of the function $f \in C(S_1)$, $P(f' - f'') = f' - f''$ implies $f' - P(f') = f'' - P(f'')$. This proves (a) and (b).

We prove (c) using the results and techniques of [4]. Let $M(S) = C(S)^*$ denote the regular Borel measures on S and for $t \in T$ let δ_t be the evaluation functional at t on $C(T)$, that is, $\delta_t(f) = f(t)$ for $f \in C(T)$. Define the linear operator $u : C(S) \to C(T)$ by $u = (\phi^0)^{-1} P$ and define $\mu : T \to M(S)$ by $\mu(t) = u^*(\delta_t)$ where u^* is the adjoint of u. Denote $\mu(t)$ by μ_t. Since $\phi^0 : C(T) \to A_\phi$ is an isometry, $\| P \| = \| u \|$. Since $u(f)(t) = u^*(\delta_t)(f) = \mu_t(f)$, $\| P \| = \| u \| = \sup\{ \| u(f) \| : \| f \| \leqq 1\} = \sup\{ | u(f)(t) | : \| f \| \leqq 1, t \in T\} = \sup\{ | \mu_t(f) | : \| f \| \leqq 1, t \in T\} = \sup\{ \| \mu_t \| : t \in T\}$. A similar argument shows that $\| P_1 \| = \sup\{ \| \mu_t \| : t \in J = \bar{P}_\phi\}$.

Since P is a projection onto $A_\phi = \phi^0(C(T))$, $u(\phi^0(f)) = f$ for $f \in C(T)$. Thus, in the terminology of [4], u is an averaging operator and $\mu : T \to M(S)$ is a dual map which averages ϕ (that is, $\mu_t(\phi^{-1}(B)) = \delta_t(B)$ for each Borel set B in T). Thus by [4, Lem. 2], if $p \in S$ and $\phi(p) = q$, then $\| \mu_q \| \leqq \sup\{ \| \mu_t \| : t \in T\} - 1 - |1 - \mu_q(\{p\})| + | \mu_q(\{p\}) |$. Suppose $q \in P_\phi$. Then $\phi^{-1}(q)$ is plural and $\mu_q(\phi^{-1}(q)) = \delta_q(\{q\}) = 1$. So there must be a point $p \in \phi^{-1}(q)$ with $- | 1 - \mu_q(\{p\}) | + | \mu_q(\{p\}) | \leqq 0$. Thus $\| \mu_q \| \leqq \sup\{ \| \mu_t \| : t \in T\} - 1 = \| P \| - 1$ for $q \in P_\phi$. If $t \in \bar{P}_\phi$ there is a net $q(\alpha) \to t$ with $q(\alpha) \in P_\phi$. Then $\mu_{q(\alpha)} \to \mu_t$ in the weak* topology on $M(S)$. Thus by [4, Lem. 1], for $t \in, \bar{P}_\phi \| \mu_t \| \leqq \liminf \| \mu_{q(\alpha)} \| \leqq \| P \| - 1$. Thus $\| P_1 \| = \sup\{ \| \mu_t \| : t \in \bar{P}_\phi\} \leqq \| P \| - 1$.

LEMMA 2.3. *Let $\phi : S \to T$ be a Gleason map. Let T_A be the Amir boundary of T and let $P_\phi = \{t \in T : \phi^{-1}(t)$ is plural$\}$. Then $T_A = \bar{P}_\phi$.*

PROOF. It suffices to show that ϕ is one-to-one on the set $\phi^{-1}(U)$ for U open in T if and only if U is extremally disconnected. Using the compactness of S, it is easy to prove that, if ϕ is one-to-one on $\phi^{-1}(U)$, then ϕ is an open map on $\phi^{-1}(U)$ (see [4, proof of Cor. 2]). Thus ϕ is a homeomorphism on $\phi^{-1}(U)$ and so U is extremely disconnected. The converse follows from a result of Gleason [6, Lem.

2.3], [15, Th. 24.2.10]: if $f: X \to Y$ is an irreducible map from one topological space onto another and Y is extremally disconnected, then f is one-to-one.

LEMMA 2.4. *Let $\phi : S \to T$ be an onto continuous map. Let $P_\phi = \{t \in T : \phi^{-1}(t)$ is plural$\}$ and let $S_1 = \phi^{-1}(\bar{P}_\phi)$. Suppose $P : C(S) \to A_\phi$ is an averaging projection for ϕ and let $E : C(S_1) \to C(S)$ be defined as in Lemma 2.2 (b). Then for each open set U in S_1 there is a function $f \in C(S_1)$ supported on U such that $E(f) \neq 0$.*

PROOF. Let U be an open set in S_1. Choose an open set W in S with $W \cap S_1 = U$. Since $S_1 = \phi^{-1}(\bar{P}_\phi) = \overline{\phi^{-1}(P_\phi)}$, there is a point $t_0 \in P_\phi$ with $\phi^{-1}(t_0) \cap W \neq \emptyset$. Since $t_0 \in P_\phi$, we can choose points s_1 and s_2 with $s_1 \neq s_2$ and $s_1 \in \phi^{-1}(t_0) \cap W$ and $\phi(s_1) = \phi(s_2)$. Choose $f' \in C(S)$ supported on W with $f'(s_1) \neq f'(s_2)$. Let $f = f' \mid S$. Then f is supported on U and we claim that $E(f) \neq 0$. To the contrary, suppose $E(f) = 0$. As defined in Lemma 2.2 (b), $E(f) = f' - P(f')$. Thus $f' = P(f')$ and so $f' \in A_\phi$. This is a contradiction since $f'(s_1) \neq f'(s_2)$.

The next lemma is closely related to [13, Th. 4.5] of Rosenthal.

LEMMA 2.5. *Suppose $E : C(X) \to C(Y)$ is a bounded linear operator where X and Y are compact Hausdorff spaces. Suppose that for every non-empty open set U in X there is an $f \in C(X)$ supported on U such that $E(f) \neq 0$. Then if Y satisfies the* CCC, *so does X.*

PROOF. (The author is indebted to the referee for this simplification of his original proof.) We use the following result due to Rosenthal [13, Lem. 4.2]. Let Y satisfy the CCC and suppose that $\mathcal{F}$ is an uncountable family of non-empty open subsets of Y. Then there is an infinite sequence $V_1, V_2, \cdots$ of distinct members of $\mathcal{F}$ with $\bigcap_{i=1}^{\infty} V_i \neq \emptyset$.

Suppose that X does not satisfy the CCC. Then there is an uncountable family $\{U_\alpha\}_{\alpha \in I}$ of disjoint non-empty open sets in X. For each $\alpha \in I$ choose $f_\alpha \in C(X)$ supported on U_α with $\|f_\alpha\| \leqq 1$ such that $E(f_\alpha) \neq 0$. The index set I is the countable union of the sets $I_n = \{\alpha \in I : \| E(f_\alpha) \| > 1/n\}$ for $n = 1, 2, \cdots$. So for some positive integer m, the set I_m is uncountable. By possibly replacing f_α by $-f_\alpha$, we may assume that for each $\alpha \in I_m$, $\sup E(f_\alpha)(y) > 1/m$. For each $\alpha \in I_m$, let $V_\alpha = \{y \in Y : E(f_\alpha)(y) > 1/m\}$. Since Y satisfies the CCC, by Rosenthal's result, there is a countable set $\{\alpha(i)\}_{i=1}^{\infty}$ with $\alpha(i) \in I_m$ such that $\bigcap_{i=1}^{\infty} V_{\alpha(i)} \neq \emptyset$. For each integer n, if $y \in \bigcap_{i=1}^{\infty} V_{\alpha(i)}$, then $E(\sum_{i=1}^{n} f_{\alpha(i)})(y) \geqq n/m$. This is impossible since the functions f_α are disjointly supported and thus $\| \sum_{i=1}^{n} f_{\alpha(i)} \| \leqq 1$.

LEMMA 2.6. *If S is an extremally disconnected compact Hausdorff space and S_1 is a closed subset of S which satisfies the* CCC, *then S_1 is extremally disconnected.*

PROOF. A topological space is an F-space or quasi-disconnected if disjoint open F_σ subsets have disjoint closures. A closed subset of an F-space is an F-space [15, Prop. 24.2.5]. Since an extremally disconnected space is an F-space, S_1 is an F-space. The lemma now follows from a result of Rosenthal [14, p. 19]: if X is a compact F-space satisfying the CCC then X is extremally disconnected.

PROOF OF PROPOSITION 2.1. Part (a) of the proposition follows immediately from Lemma 2.2 (a) and (c) and from Lemma 2.3. To prove part (b), Lemma 2.4 shows that the operator E defined in Lemma 2.2 (b) satisfies the hypothesis of Lemma 2.5. Thus if S satisfies the CCC then so does S_1 and Lemma 2.6 implies that S_1 must then be extremally disconnected.

3. Proof of main theorem and corollaries

PROOF OF THEOREM 1.1. Suppose $C(T)$ is a P_λ-space and let $\phi : S \to T$ be a Gleason map for T. Then there is an averaging projection P for ϕ with $\| P \| \leqq \lambda$. Let $S_1 = \phi^{-1}(T_A)$ and let $\phi_1 = \phi | S_1 : S_1 \to T_A$. By Proposition 2.1 (a), there is an averaging projection P_1 for ϕ_1 with $\| P_1 \| \leqq \| P \| - 1 \leqq \lambda - 1$. By Proposition 2.1 (b), S_1 is extremally disconnected and so $C(S_1)$ is a P_1-space. Thus $C(T_A)$ is isometric to the space A_{ϕ_1} which is complemented in the P_1-space $C(S_1)$ by a projection of norm less than or equal to $\lambda - 1$. Therefore $C(T_A)$ is a $P_{\lambda-1}$-space. By Proposition 2.1 (b), S_1 satisfies the CCC and therefore T_A satisfies the CCC since $\phi : S_1 \to T_A$ is onto.

PROOF OF COROLLARY 1.2. This corollary is immediate since if $C(T_A)$ is a P_λ-space for $\lambda < 2$, then T_A is extremally disconnected (Amir [1, Cor. a], Isbell and Semadeni [8, Th. 1 (ii)].

PROOF OF COROLLARY 1.3. Let $T_0 = T$ and $T_1 = T_A$. Then $T_0 - T_1$ is an open dense extremally disconnected subset of T_0. By Theorem 1.1, $C(T_A) = C(T_1)$ is a $P_{\lambda-1}$-space and $T_1 = T_A$ satisfies the CCC. Define T_2 to be the Amir boundary of T_1. Applying Theorem 1.1 again we see that T_2 satisfies the CCC and $C(T_2)$ is a $P_{\lambda-2}$-space. Also, $T_1 - T_2$ is an open dense extremally disconnected subset of T_1. In general, define T_i to be the Amir boundary of T_{i-1}. Since the value of λ decreases by at least 1 at each step, we eventually end up with a set T_n which is a P_λ-space for $\lambda < 2$. So T_n is extremally disconnected.

Corollary 1.4 follows trivially from Corollary 1.3 and the following topologica lemma.

LEMMA 3.1. (The author is indebted to the referee for this generalization andl proof of his original lemma.) *Suppose S is a compact Hausdorff space and $S = U \cup B$ where U is open. If U and B are totally disconnected, then so is S.*

PROOF. We may assume $U \cap B = \emptyset$ so that B is closed. Given $s_0 \in G \subset S$, G open, we look for a closed and open set W with $s_0 \in W \subset G$. If s_0 is in U this is immediate. If $s_0 \in B$ let A be a closed and open set in B with $s_0 \in A \subset G$. Let $F: S \to [0,1]$ be a continuous function such that $F(A) = 0$ and $F(\{B \setminus A\} \cup \{S \setminus G\}) = 1$. Since $F(B) = \{0,1\}$, $F^{-1}(\frac{1}{2})$ is a compact subset of U. Let K be a closed and open set with $F^{-1}(\frac{1}{2}) \subset K \subset G \cap U$. Let $W = K \cup \{S : f(s) < \frac{1}{2}\}$. Clearly W is open and $s_0 \in W \subset G$. Also W is closed since if (s_α) is a net with $\lim s_\alpha = s$ then eventually $F(s_\alpha) < \frac{1}{2}$ or $s \in F^{-1}(\frac{1}{2}) \subset K$.

4. Remarks and open problems

The compact Hausdorff spaces satisfying the CCC include all separable spaces and all spaces which support a measure, that is, there is a finite regular Borel measure μ with $\mu(U) \neq 0$ for each open set U. Included in this last class of spaces are all spaces T such that there is an onto map $\phi : S \to T$ where S is the maximal ideal space of a Banach algebra $L^\infty(\mu)$ for μ a finite measure. The Stone-Cech compactification $B(\Gamma)$ of an uncoutable discrete set Γ does not satisfy the CCC.

It seems to be open whether Theorem 1.1 and its corollaries remain true without assuming T satisfies the CCC.

PROBLEM 4.1. Suppose $C(T)$ is a P_λ-space and T does not satisfy the CCC. Must $C(T_A)$ be a $P_{\lambda-1}$-space where T_A is the Amir boundary of T?

The techniques of this paper show that a positive solution to this problem is implied by a positive solution to the following problem.

PROBLEM 4.2. Suppose $\phi : S \to T$ is a Gleason map and $C(T)$ is a P_λ-space. Suppose T does not satisfy the CCC. Must $\phi^{-1}(T_A)$ be extremally disconnected?

A positive solution to either of the above problems implies a positive solution to the next problem.

PROBLEM 4.3. Suppose $C(T)$ is injective. Must $C(T_A)$ be injective?

Assuming the continuum hypothesis, [14, Cor. 6] of Rosenthal can be used to show that the three problems are equivalent for spaces T whose topological weight is equal to the continuum.

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